LIMIT POINTS OF A SEQUENCE :-

Let $\langle a_n \rangle$ be a real sequence, a real number ξ is said to be a limit point of the sequence if for every given $\epsilon > 0$,

$|a_n - \xi| < \epsilon$, for infinitely many values of n.

i.e., $|a_n - \xi| < \epsilon$, for infinitely many n

$\Rightarrow -\epsilon < a_n - \xi < \epsilon$,

$\Rightarrow \xi - \epsilon < a_n < \xi + \epsilon$

$\Rightarrow a_n \in (\xi - \epsilon, \xi + \epsilon)$, for infinitely many n.

Example:- 1) $\langle a_n \rangle = \langle 1 \rangle$, $n \in N$, has the only limit point 1. The range is $\{1\}$ and has no limit point.

2) $\langle a_n \rangle = \langle \frac{1}{n} \rangle$, $n \in N$, has 0 as a limit point as well a limit point of the range $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

3) i) $\langle a_n \rangle = \langle 1 + (-1)^n \rangle$, $n \in N$ ii) $\langle a_n \rangle = \langle (-1)^n \rangle$, $n \in N$

iii) $\langle a_n \rangle = \langle (-1)^n (1 + \frac{1}{n}) \rangle$, $n \in N$, Explain ?

Now from ① and ②, for $n \geq \max(m_1, m_2)$

$$|l - l'| = |l - a_n + a_n - l'| \leq |l - a_n| + |a_n - l'| < 2\epsilon$$

i.e., $|l - l'| < \frac{2}{3}|l - l'|$, which is not possible

Hence the sequence cannot converge to two limits.

NOTE:
Archimedean Property: If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ s.t. $x < n_x$.

Example:- 1) Show that $\lim \left(\frac{1}{n^2+1}\right) = 0$

Proof:- Let ϵ be any positive number

$$\therefore \left|\frac{1}{n^2+1} - 0\right| < \left|\frac{1}{n^2}\right| \leq \left|\frac{1}{n}\right| < \epsilon, \text{ or } n > \frac{1}{\epsilon}$$

Let m be a positive integer greater than $\frac{1}{\epsilon}$

Thus to $\epsilon > 0$, $\exists$ a positive integer m, s.t.

$$\left|\frac{1}{n^2+1} - 0\right| < \epsilon, \forall n \geq m$$

$$\therefore \lim \left(\frac{1}{n^2+1}\right) = 0$$

Example: 2) Show that $\lim\limits_{n \to \infty} \frac{3 + 2\sqrt{n}}{\sqrt{n}} = 2$.

For, if possible $\langle a_n \rangle = \langle (-1)^n \rangle, n \in N$ is convergent

i.e $\lim_{n\to\infty} a_n = l$ then for $\epsilon = 1, \exists\, m \in N$,

s.t $\quad |a_n - l| < 1, \forall\; n \geq m$ i.e.,

$|(-1)^{2m} - l| < 1$ and $|(-1)^{2m+1} - l| < 1$

or $|1 - l| < 1$ and $|-(1+l)| = |1+l| < 1$

$\Rightarrow 2 = |1-l+1+l| < 1+1 = 2$

$\Rightarrow 2 < 2$, this is not possible (or absurd)

So. $\langle a_n \rangle = \langle (-1)^n \rangle, n \in N$ is not convergent.

Theorem 2. A sequence cannot converge to more than one limit. (Uniqueness of limits)

Proof:- Let, if possible, a sequence $\langle a_n \rangle$ converge to two real numbers l and l' $(l \neq l')$. Let us select $\epsilon = \frac{1}{3}|l - l'| > 0$

Since the sequence $\langle a_n \rangle$ converges to l and l', therefore, there exist positive integer m_1, m_2

Such that $|a_n - l| < \epsilon \quad \forall\; n \geq m_1$ —— (1)

and $|a_n - l'| < \epsilon \quad \forall\; n \geq m_2$ —— (2)

NOTE: $|x| < \epsilon \Leftrightarrow -\epsilon < x < \epsilon$.

NOTE: If a sequence has a limit, we say that the sequence is convergent; if it has no limit, we say that the sequence is divergent.

SOME THEOREMS:-

THEOREM 1. Every convergent sequence is bounded.

Proof:- Let a sequence $\langle a_n \rangle$ converges to the limit 'l'.

let $\epsilon > 0$ be a given number, so that $\exists$ a positive integer m such that

$$|a_n - l| < \epsilon \quad \forall \ n \geq m$$

$$\Leftrightarrow \quad l - \epsilon < a_n < l + \epsilon, \quad \forall \ n \geq m$$

let

$$g = \min\{l-\epsilon, a_1, a_2, \ldots\ldots, a_{m-1}\}$$

$$G = \max\{l+\epsilon, a_1, a_2, \ldots\ldots, a_{m-1}\}$$

Thus we have $g \leq a_n \leq G$, $\forall \ n$

Hence $\langle a_n \rangle$ is a bounded sequence.

Remark:- The converse of the above theorem may not be true. For example the sequence $\langle a_n \rangle$, where $a_n = (-1)^n$, $n \in N$, is bounded but it is not convergent.